Math 208 Midterm 1

Jan 24th, 2024

Name______________________________ **Section number**______

There are 5 problems, which are 35 points in total.

- (Academic honesty) **Sign your name below:**
  " I have not given or received any unauthorized help on this exam"
  [signature] ____________________
- You are allowed to use a summary sheet.
- No other resources are allowed (Internet, graphing calculator, other humans, . . .).
- **All answers must be justified or with necessary steps. You will receive at most** 1 **point for an answer without any explanation.**
- Please write down your initials **in each page**. We will scan your exams and upload to gradescope.
- **DO NOT WRITE ON THE BACK OF EACH PAGE.** Instead, use the last page for extra space.

Initials ____________________

Problem 1.(8 points) Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$, where A is the following matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(1)(2 points) Find the values of m and n.

(2)(3 points) Is the matrix A in the reduced echelon form? If not, convert it into the reduced echelon form.

(3)(3 points) Is the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the range of T? If so, find a vector $\mathbf{x}$ in $\mathbb{R}^m$ such that $T(\mathbf{x}) = \mathbf{b}$.

Initials ___________________

Problem 2. (8 points) Consider the following vectors.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix},$$

(1)(3 points) Are $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ linearly independent? Explain why.

(2)(2 points) Does $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ span $\mathbb{R}^3$? Explain why.

(3)(3 points) Is the vector $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ in the span of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$? Write down necessary steps to justify your answer.

Initials ___________________

Problem 3. (7 points) For each of the following, find an example if possible and explain your example. Or explain why such an example cannot exist.

(1) (3 points) Find, if possible, three linearly independent vectors in the plane $x + y - z = 0$.

(2) (4 points) Find, if possible, a set of vectors in $\mathbb{R}^3$ that span $\mathbb{R}^3$ but are linearly dependent.

Initials ___________________

Problem 4. (8 points)

(1) (4 points) Consider the following vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -8 \\ z_1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ -2 \\ z_2 \end{bmatrix}.$$

Find all values of z_1, z_2 such that the vector equation $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{b}$ has a unique solution.

(2) (4 points) Find a 3×4 matrix A, in *reduced* echelon form, such that the general solution of the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

where s is any real numbers.

Problem 5. (4 points) For the following statement, either find an example that contradicts the statement to show that it is false, or explain why the statement is always true.

If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a spanning set for $\mathbb{R}^n$, then $\{\mathbf{u}_1 + 2\mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3, \mathbf{u}_2 - \frac{1}{2}\mathbf{u}_3\}$ also spans $\mathbb{R}^n$.